



Design and Implementation of New Biorthogonal Wavelets and its Application to Image Processing

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Abstract— Wavelet transformation has been an interesting field since its exposure with wavelet-based compression standard embedded zerotree wavelet. Though, the origin of wavelets back to many decades, the presence of research at the very beginning of wavelet bases is also being carried out in the research community. This is because of the wide presence of its applicability as well as its structure of adapting to the type of problem at hand. In this paper, an approach of designing biorthogonal wavelets is presented. This approach may be extended to design variety of wavelets. This paper mainly focus on the design steps and corresponding coding in MATLAB. By varying the length, type of the bases and other parameters, different wavelets may be devised.

Index Terms— Biorthogonal wavelets, Spline function, wavelet base, wavelet design

1 ORIGIN OF THE PROPOSAL

Wavelet transformation has extended its presence from mathematical to signal processing, image analysis and industrial applications. In contrast to its predecessors, it has the ability of adapting to the application as well as input signal at hand. In Fourier transform, there is no flexibility of selecting basis function and it is not possible to resolute in time domain. The short-time Fourier transform, however, has the feature of providing some sort of time information along with the frequency data at these intervals. The wavelet transform, on the other hand, gives a complete control over both the time and frequency intervals.

As mentioned earlier, in addition to the flexibility of time-frequency resolution, the wavelet transform has a feature selecting different basis functions for different problems. The definition of wavelet transform itself has the feature selecting basis function, which is fixed and can't be changed in earlier Fourier transform based schemes.

This flexibility has opened the doors of designing new wavelets for different applications and for different input categories. The very beginning of wavelet transform started by the definition of Haar function which is used as basis function of the very first wavelet, i.e., Haar wavelet. Since then, as many as thousands of wavelets are designed, implemented, and applied on different problems. By changing the parameters of one basis function itself, many wavelets are designed as in Daubechies wavelets. In this paper, design procedure of bioorthogonal wavelets is presented.

In 1994, R. A. DeVore proposed a flexible foundation for image compression [1]. In 1996, Michael G. Strintzis proposed biorthogonal wavelet bases for signal decay [2]. In 2007, Kharate, Patil and Bhale compared the image code application using different wavelet bases [3]. In 2013, Vijay S. Chourasia and Anil Kumar Tiwari proposed a novel algorithm based on wavelet mutations developed to extract signals from Fetal phonocardiography (fPCG) [4]. In 2015, Noor Kamal, Mohd Ali, Ahmad, Shabiul Islam and Escudero proposed a selection of motherboard EEG signal analysis for multiple channels during the working memory process [5]. In 2016, Gengesa Garg proposed an algorithm for selecting a fixed wave function [6].

2 IMPORTANCE OF THE PROPOSAL IN THE CONTEXT OF CURRENT STATUS

In all the applications, the wavelet is treated as a tool to represent the data in a representation where the specific application can be served in an efficient way. The algorithms associated with each application are run on the wavelet transformed data. This data is a result of inner product of original time/spatial domain data with the basis function of the wavelet. The inner product serves as the similarity measure between the input data and basis function. In inverse wavelet transform, another inner product will result a function which has such similarity with the basis functions. All the standard wavelets will satisfy the perfect reconstruction where the reconstructed signal is the same as that at input. It's not just the perfect reconstruction which has to be met, but also the recital of concerned application.

3 WAVELETS IN IMAGE PROCESSING

Few decades ago, there are finite and defined image processing tasks that are carried and experimented by the research community of image processing and computer vision. But, over the past decade there are many distinct and custom image processing tasks introduced to serve the purpose of the minute. In this section, few of traditional and modern applications of image processing are review. It is important to know that the wavelet processing has influenced if not all, most of the image processing tasks.

The first one to mention is image compression. Image compression involves transformation of raw data into a form which is more compatible to a small memory without losing much information. Early image compression algorithms are implemented in spatial domain where the pixel values are coded in some fashion to incorporate compression. But, in 1993, a famous image compression codec is proposed which has changed the fate of wavelets [7]. The codec is Embedded zerotree wavelet. Coding was done by transferring significant map of wavelet domain in a systematically designed symbol format. Further, these symbols are coded using regular coding schemes like entropy coding. Later, in 1996, another popular image compression standard Set partitioning in hierarchical trees (SPIHT) and then many advanced coding schemes are proposed for wavelet-based image compression [8]-[11].

Image restoration is another key task that is being performed using wavelet transforms. Image restoration can be deblurring, denoising, compressed sensing or super resolution [12]. Image restoration involves a sequence of steps to restore the best estimate of the original image from deteriorated version of the image. The next generation schemes like sparse coding are evolved from wavelet usage extensions in image restoration [13]. Wavelet domain was analyzed in depth to understand what the wavelet domain coefficients contains and how they relate and affect the reconstruction process. Wavelet coefficients themselves are adaptive in default form, yet the sparse coding was derived from this kind of analysis.

Image watermarking is extensively implemented in wavelet domain. The clue for this is that there exist multiple levels as decomposition can be extended in hierarchical structure. Also, out of the decomposition bands, few are of less importance. This gives rise to use the band for storing secret information without changing the host appearance [14].

Image fusion is task of mixing multiple images into one image. This is generally done when the object of interest possesses different appearance when viewed in different angles. Normally, multi-sensor system requires image fusion by default. As mentioned earlier, wavelet domain has multiple bands, and these bands are mixed in different ways to produce fused image [15].

Some modern applications of wavelets in image processing involve corner detection, deraining, dehazing, object counting and many more [16][17]. As there exists many areas in industry where human interventions are difficult, autonomous systems are employed where images of the scene are processed in real-time to extract whatever the information we need in no time.

4 METHODOLOGY

The design process of wavelet differs by its type. Fundamentally, there are two types of wavelets: orthogonal and biorthogonal. As the biorthogonal wavelets has an additional advantage of design flexibility, these wavelets will be chosen for further research.

$$\text{Since} \quad \int \phi(t) \tilde{\phi}(t-k) dt = \delta_{k,0}$$

$$\sum_n h(n) \tilde{h}(n-2k) = \delta_{k,0} \quad (1)$$

i.e., $h(n)$ is orthogonal to even translates of itself. Here \tilde{h} is orthogonal to h . Now let the positioning of $\phi(t)$ and $\tilde{\phi}(t)$ is from N_1 to N_2 and \tilde{N}_1 to \tilde{N}_2 respectively. Equation (1) implies

$$N_2 - \tilde{N}_1 = 2p + 1 \text{ and } \tilde{N}_2 - N_1 = 2\tilde{p} + 1, \quad p, \tilde{p} \in Z \quad (2)$$

If a wrong position is chosen, there will be the chance of getting all coefficients with zero value. In addition, the positioning directly constrains the coefficients. If equation (2) is not satisfied there will be no feasible solution for $\tilde{h}(k)$ coefficients. One of such position with which a feasible design is made is given below.

$$N_1 = 1, \quad N_2 = 5 \text{ and } \tilde{N}_1 = 0, \quad \tilde{N}_2 = 6 \quad (3)$$

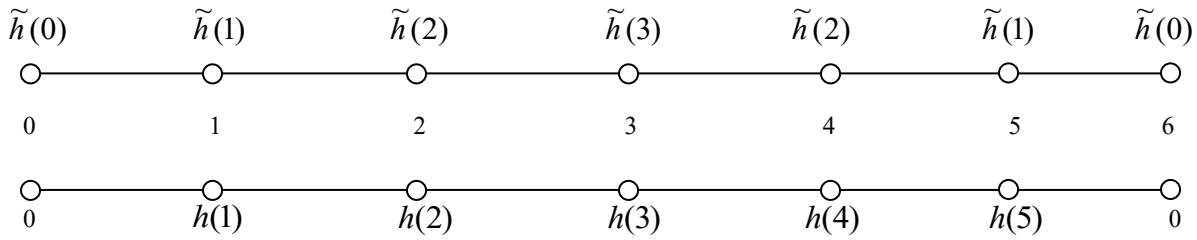
Here, several other possibilities are also there. In fact, the selection can be made in infinite ways. But, as the size is less, they approximate to existing wavelets and when the size is large, the performance will not be good. The reason for this is, the dilation will lose its meaning when the basis function is already stretched. Many applications require symmetric scaling function coefficients. So let us put this constraint first. Hence,

$$\tilde{h}(0) = \tilde{h}(6), \tilde{h}(1) = \tilde{h}(5) \text{ and } \tilde{h}(2) = \tilde{h}(4) \quad (4)$$

The equation $\int \tilde{\phi}(t) dt = 1$ results in $\sum_k \tilde{h}(k) = \sqrt{2}$,

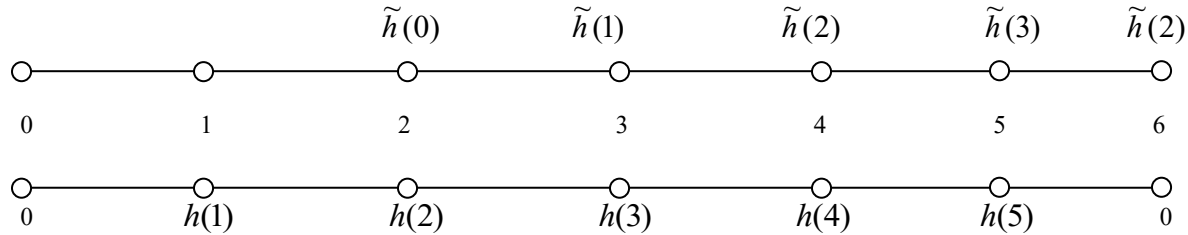
$$\text{hence} \quad 2\tilde{h}(0) + 2\tilde{h}(1) + 2\tilde{h}(2) + \tilde{h}(3) = \sqrt{2} \quad (5)$$

Now use equation (1), i.e., $\sum_n h(n) \tilde{h}(n-2k) = \delta_{k,0}$. Substituting $k = 0$ in (1),



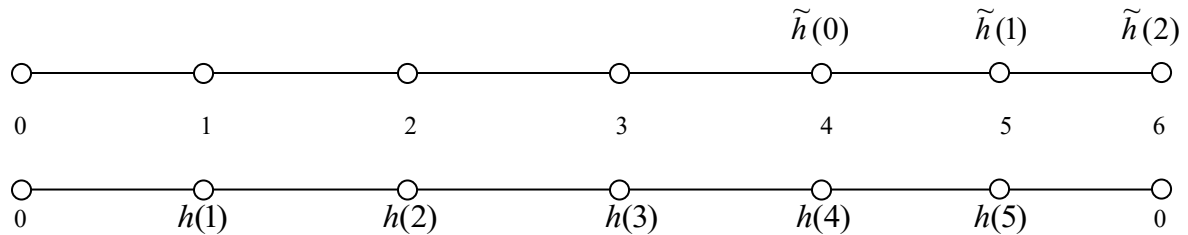
$$h(1)\tilde{h}(1) + h(2)\tilde{h}(2) + h(3)\tilde{h}(3) + h(4)\tilde{h}(2) + h(5)\tilde{h}(1) = 1 \tag{6}$$

Substituting k = 1 in $\sum_n h(n) \tilde{h}(n - 2k) = \delta_{k,0}$,



$$h(2)\tilde{h}(0) + h(3)\tilde{h}(1) + h(4)\tilde{h}(2) + h(5)\tilde{h}(3) = 0 \tag{7}$$

Substituting k = 2 in $\sum_n h(n) \tilde{h}(n - 2k) = \delta_{k,0}$,



$$h(4)\tilde{h}(0) + h(5)\tilde{h}(1) = 0 \tag{8}$$

The general form of Spline of order k is given by

$$N_k(t) = \sum_{i=0}^k p_i N_k(2t - i)$$

$$p_i = \frac{1}{2^{k-1}} \binom{k}{i}$$

where

Consider Spline of order 4 which is given below.

$$N_4(t) = \frac{1}{8} N_4(2t) + \frac{4}{8} N_4(2t-1) + \frac{6}{8} N_4(2t-2) + \frac{4}{8} N_4(2t-3) + \frac{1}{8} N_4(2t-4)$$

The above spline is considered as one of the scaling functions. Hence the un-normalized coefficients become, $\frac{1}{8}, \frac{4}{8}, \frac{6}{8}, \frac{4}{8}$ and $\frac{1}{8}$. The sum of normalized coefficients must be equal to $\sqrt{2}$ this follows from the requirement

$\int \phi(t) dt = 1$. Therefore $a\left(\frac{1}{8} + \frac{4}{8} + \frac{6}{8} + \frac{4}{8} + \frac{1}{8}\right) = \sqrt{2} \Rightarrow a = \frac{1}{\sqrt{2}}$. Hence the normalized coefficients of $\phi(t)$ are:

$$\frac{1}{8\sqrt{2}}, \frac{4}{8\sqrt{2}}, \frac{6}{8\sqrt{2}}, \frac{4}{8\sqrt{2}} \text{ and } \frac{1}{8\sqrt{2}}.$$

Using equation (3), the scaling coefficients becomes

$$h(1) = \frac{1}{8\sqrt{2}}, h(2) = \frac{4}{8\sqrt{2}}, h(3) = \frac{6}{8\sqrt{2}}, h(4) = \frac{4}{8\sqrt{2}} \text{ and } h(5) = \frac{1}{8\sqrt{2}} \tag{9}$$

Rewriting the above equations using (9) becomes

$$\text{Eq. (5)} \rightarrow 2\tilde{h}(0) + 2\tilde{h}(1) + 2\tilde{h}(2) + \tilde{h}(3) = \sqrt{2} \tag{10}$$

$$\text{Eq. (6)} \rightarrow \frac{1}{4\sqrt{2}}\tilde{h}(1) + \frac{1}{\sqrt{2}}\tilde{h}(2) + \frac{3}{4\sqrt{2}}\tilde{h}(3) = 1 \tag{11}$$

$$\text{Eq. (7)} \rightarrow \frac{1}{2\sqrt{2}}\tilde{h}(0) + \frac{3}{4\sqrt{2}}\tilde{h}(1) + \frac{1}{2\sqrt{2}}\tilde{h}(2) + \frac{1}{8\sqrt{2}}\tilde{h}(3) = 0 \tag{12}$$

$$\text{Eq. (8)} \rightarrow \frac{1}{2\sqrt{2}}\tilde{h}(0) + \frac{1}{8\sqrt{2}}\tilde{h}(1) = 0 \tag{13}$$

For a total of 4 variables, four equations are formed but, in these equations, only the equations (10), (11) and (13) are independent. Hence another equation is formed below using vanishing moments condition,

$$\tilde{h}(6) - \tilde{h}(5) + \tilde{h}(4) - \tilde{h}(3) + \tilde{h}(2) - \tilde{h}(1) + \tilde{h}(0) = 0. \text{ This implies} \tag{14}$$

$$2\tilde{h}(0) - 2\tilde{h}(1) + 2\tilde{h}(2) - \tilde{h}(3) = 0$$

Solving (10), (11), (13) and (14) yields

$$\tilde{h}(0) = \frac{3\sqrt{2}}{32}, \tilde{h}(1) = -\frac{3\sqrt{2}}{8}, \tilde{h}(2) = \frac{5\sqrt{2}}{32} \text{ and } \tilde{h}(3) = \frac{5\sqrt{2}}{4}.$$

By symmetry

$$\tilde{h}(4) = \frac{5\sqrt{2}}{32}, \tilde{h}(5) = -\frac{3\sqrt{2}}{8} \text{ and } \tilde{h}(6) = \frac{3\sqrt{2}}{32}.$$

Since $\tilde{\phi}(t) \perp \psi(t)$, $g(k) = (-1)^k \tilde{h}(N - k - 1)$. Therefore

$$g(-1) = -\frac{3\sqrt{2}}{32}, g(0) = -\frac{3\sqrt{2}}{8}, g(1) = -\frac{5\sqrt{2}}{32}, g(2) = \frac{5\sqrt{2}}{4}, g(3) = -\frac{5\sqrt{2}}{32},$$

$$g(4) = -\frac{3\sqrt{2}}{8} \text{ and } g(5) = -\frac{3\sqrt{2}}{32}.$$

Similarly using $\tilde{g}(k) = (-1)^k h(M - k - 1)$

$$\tilde{g}(0) = \frac{1}{8\sqrt{2}}, \tilde{g}(1) = -\frac{4}{8\sqrt{2}}, \tilde{g}(2) = \frac{6}{8\sqrt{2}}, g(3) = -\frac{4}{8\sqrt{2}}, \text{ and } g(4) = \frac{1}{8\sqrt{2}}.$$

The coefficients of newly designed biorthogonal wavelet are given in the table below.

K	-01	00	01	02	03	04	05	06
\tilde{h}		0.13260	-0.53030	0.22100	1.76780	0.22100	-0.53030	0.13260
\tilde{g}		0.08840	-0.35360	0.53030	-0.35360	0.08840		
h			0.0884	0.3536	0.5303	0.3536	0.0884	
g	-0.1326	-0.5303	-0.2210	1.7678	-0.2210	-0.5303	-0.1326	

Simulation Work:

Solving the equations.

`syms h0 h1 h2 h3;`

```
[h0 h1 h2 h3] = solve('2*h0 + 2*h1 + 2*h2 + h3 - sqrt(2)',...
    '(1/(4*sqrt(2)))*h1 + (1/sqrt(2))*h2 + (3/(4*sqrt(2)))*h3 -1',...
    '(1/(2*sqrt(2)))*h0 + (1/(8*sqrt(2)))*h1',...
    '2*h0 - 2*h1 + 2*h2 - h3')
```

2. Adding the designed wavelet to the toolbox.

`familyName = 'New Biorthogonal of JK';`

`familyShortName = 'jkb';`

`familyWaveType = 2;`

`familyNums = '';`

`fileWaveName = 'jkb.mat';`

`Df = [(1/(8*sqrt(2))), (4/(8*sqrt(2))), (6/(8*sqrt(2))), ...`
`(4/(8*sqrt(2))), (1/(8*sqrt(2)))];`

`Rf = [(3*sqrt(2)/32), (-3*sqrt(2)/8), (5*sqrt(2)/32), ...`
`(5*sqrt(2)/4), (5*sqrt(2)/32), (-3*sqrt(2)/8), (3*sqrt(2)/32)]`

```

save jkb Rf Df
wavemngr('add',familyName,familyShortName,familyWaveType, ...
    familyNums,fileWaveName)
wavemngr('read')

```

3. Plotting the wavelet functions.

```

wname = 'jkb';
clf; wavefun(wname,'plot',7);

```

4. Checking the Perfect Reconstruction on Images

```

I = imread('lena.jpg');
[cA,cH,cV,cD] = dwt2(I,'jkb');
X = idwt2(cA,cH,cV,cD,'jkb');
[PSNR,MSE] = measerr(I,uint8(X))

```

5. Apply for Image Compression

```

I = imread('lena.jpg');
[cA,cH,cV,cD] = dwt2(I,'nbm');
dec=[...
    cA,cH
    cV,cD
    ...
];
sX=size(cA);
[r c]=size(I);
input_size=r*c*8;
bitbudget = 1000000;
[encoded bits] = cSPIHT(dec, 1, bitbudget);
CR=input_size/bits
[decoded level] = dSPIHT(encoded, bits);
sX1=size(decoded);
m1 = decoded;
CA = m1(1:(sX(1)), 1:(sX(1)));%LLH3
CH = m1(1:(sX(1)), (sX(1) + 1):sX1(1));%LHL4
CV = m1((sX(1) + 1):sX1(1), 1:(sX(1)));%HLH3
CD = m1((sX(1) + 1):sX1(1), (sX(1) + 1):sX1(1));%HHH3
Ir=idwt2(CA, CH, CV, CD,'nbm');
[PSNR] = measerr(I, Ir)

```

The above description gives the overall structure of designing a biorthogonal wavelet given the size and basis function.

5 CONCLUSIONS

The application of wavelets extended over varied fields including design, industry, and scientific analysis. The input invariant transformations are gaining importance as the new transformations are introduced. This trend is increasing with the evolution of sparse coding. Towards this end, in this paper, an attempt has been made to present the design of biorthogonal wavelets from scratch. The basis function is defined using which the scaling and wavelet coefficients are calculated. Finally, an approach to test its perfect reconstruction condition as well as application on images is presented.

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ETHICS STATEMENT

This study did not involve human or animal subjects and, therefore, did not require ethical approval.

STATEMENT OF CONFLICT OF INTERESTS

The authors declare no conflicts of interest related to this study.

LICENSING

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